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Redundant Reactions of Indeterminate Beams by Principle of Quasi Work

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A new methodology for determining redundant reactions of beams is presented in this paper. This methodology is based on the principle of quasi work, which is a powerful pseudoenergy theorem deduced from Tellegen's theorem. It takes topography of a structural system as a variable in addition to the variables involved in conventional energy methods, variational principles, and finite element methods. The concept of topologically similar system lies at the heart of the principle of quasi work. This concept is explored for beams to define topologically similar beams and topologically equivalent beams. The principle of quasi work is validated for beams and subsequently used for determining redundant reactions of indeterminate and continuous beams. Further, a unique concept of reference beam is developed. Equation of deflection curve of this reference beam is used to solve redundant reactions of indeterminate beams. This methodology has an advantage of calculating redundant reactions mostly by simple multiplications without any integrations or differentiations and does not require any prior knowledge of writing bending moment expressions. The method is illustrated through examples. It is possible to develop an interactive graphic computer package for calculating reactions of indeterminate beams.

Nomenclature

A_n	=	cross-section area of beam represented by TSS_n
$\{d\}_n$	=	displacement in TSS_n corresponding to $\{P\}_m$
E_n	=	Young's modulus of elasticity of beam represented by TSS_n
$\{F\}_m$	=	internal forces in a TSS_m
L_n	=	length of beam represented by TSS_n
M_n	=	bending moment in the beam represented by TSS_n
M_n^*	=	mapped bending moment in the beam represented by TSS_n
M_*	=	moment reaction on beam from support $*$, in which $*$ represents A, B, C, etc.
$\{P\}_m$	=	external loads acting on TSS_m
R_*	=	reaction on beam from support $*$, in which $*$ represents A, B, C, and 0, 1, 2, etc.
U_{mn}	=	quasi energy $= \{F\}_m^T\{\delta\}_n$
W_{mn}	=	quasi energy $= \{P\}_m^T\{d\}_n$
$v(x)$	=	beam deflection as a function of x
w	=	intensity of uniformly distributed load
x_n	=	coordinate along longitudinal axis of beam represented by TSS_n
y	=	coordinate along the depth of beam measured from neutral axis
$\{\delta\}_n$	=	deformation in TSS_n corresponding to $\{F\}_m$
$\varepsilon(x)$	=	strain as a function of x in the beam
$\sigma(x)$	=	stress as a function of x in the beam
ζ	=	nondimensional parameter $= x_n/L_n$
m, n	=	subscripts 1,2,3, etc., representing different topologically similar systems
$\langle(\exp.)\rangle$	=	if $(\exp.) \leq 0$ the value of the bracket $= 0$; otherwise, it is equal to $(\exp.)$
$\{(\exp.)\}_n$	=	$(\exp.)$ is for TSS_n

I. Introduction

VARIOUS methods exist for analysis of indeterminate structures. The widely used are finite element method and formulations that use either the stiffness method or the force method or a combination of these two methods. There have been attempts to solve structural mechanics problems by formulating these with the tools from other fields of science. Fenves and Branin [1] established the relationship between graph theory and force method as well as stiffness method. Sebastian [2] used electric network analogy for elastic structures. Shai [3] used graph theory and Tellegen's theorem [4,5] for deriving well-known existing structural mechanics theorems. However, Shai [3] used Tellegen's energy conservation theorem applicable for single networks. Panditta [6] also used Tellegen's theorem in the realm of structural mechanics, but, as given by Penfield et al. [7], connected two electrical networks and derived the principle of quasi work (PQW) for establishing a connection between two structural models that are topologically similar. The existing energy methods [8,9], variational principles [10,11], and finite element methods [12,13] are meant for a single system and do not provide any connection to two structural systems. Panditta et al. [14] derived theorems useful for discrete structural models characterizing stiffness, mass, and damping properties of structural elements forming the lower end of finite elements by using this powerful quasi energy theorem. Munshi and Tikoo [15] developed a demonstrative interactive computer graphic program for obtaining redundant reaction of single degree indeterminate beams.

In this paper, PQW is applied to beams. Conditions for topological similarity for the beams are derived. The concept of topologically similar system (TSS) for discrete models, truss structure, and beams is illustrated with figures. PQW is validated and subsequently illustrated with examples to bring out the outstanding generality of PQW over conventional energy methods. A unique concept of reference beam (RB) is developed and subsequently, with the help of the solution for deflection of its elastic line, indeterminate beams with

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different boundary conditions are solved for redundant reactions. The RB can be used for solving redundant reactions of any type of beam. Hence, this paves the way for development of a general purpose interactive computer program for beams.

II. Principal of Quasi Work

PQW states that in a pair of TSS (say m and n), quasi work done by (self-equilibrating system of) external forces of any one of the systems (say m) while going through the corresponding (compatible) displacements of the other system (n) is equal to the quasi-strain energy due to internal forces of former system (m) while going through corresponding deformations of the latter system (n).

In mathematical form it is stated as follows:

$$W_{mn} = U_{mn} \tag{1}$$

In the statement of PQW, topology for discrete structural models (as given by Panditta [6] and Panditta et al. [14]) specifies a unique set of nodes and their interconnectivity in a system. For TSS, the physical/material properties and the constitutive relations of branch elements (connecting different nodes) are immaterial. Boundary conditions and external loading on the system are not included in the concept of topology. Hence, for a given structural system an infinite number of TSS can be generated by varying a branch/element parameter from limiting values of zero/infinity (making such load paths vanish/rigid) and also by substituting one type of element by another type of element (e.g., replacing spring by a damper).

A typical illustration of TSS for a discrete structural model (TSS_1 and TSS_2) having the common topological layout (Fig. 1a) are shown in Figs. 1b and 1c. The concept of TSS for continuous truss structure is illustrated in Fig. 2. In this figure some of the TSS (Figs. 2b–2j) having the same topology (Fig. 2a) are included and are called topologically similar trusses (TST). In these TST dotted lines represent zero stiffness elements and bold lines represent rigid elements. Truss element(s) can be removed from the given indeterminate truss (by assigning a zero value to its/their stiffness) as long as the truss remains internally stable. This makes it possible to derive an internally determinate TST from a given indeterminate truss and external indeterminacy is removed by appropriate choice of supports. By varying boundary conditions and external loading, an innumerable number of TST can be generated. The TSS concept for beams is illustrated in Fig. 3. All the beams given in Figs. 3a–3h are topologically similar and are called topologically similar beam (TSB).

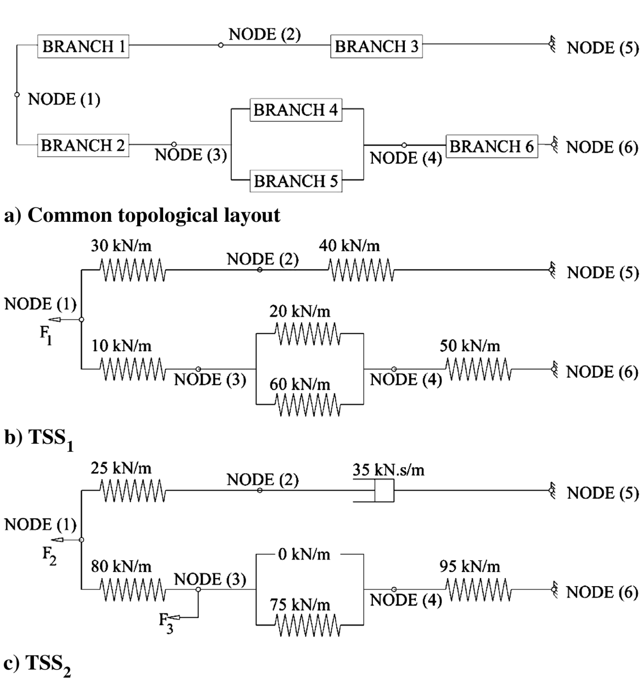

Fig. 1 TSS: discrete models.

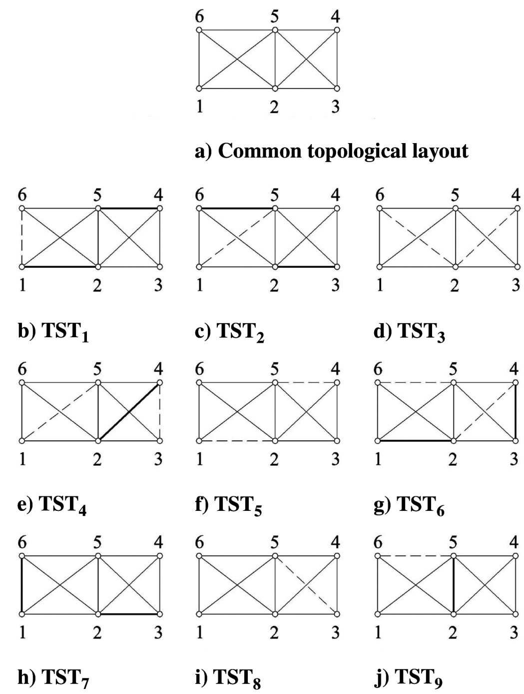

Fig. 2 TSS with common layout.

III. Principle of Quasi Work Applied to Beams

Hereunder, the concept of TSS for beams is arrived at by applying the definition of PQW to two different beams. As PQW refers to self-equilibrating systems, hence, supports will not be a part of the system instead reactions coming from supports will form a part of the external loading. Take two beam elements with parameters (L_1, A_1, E_1) and (L_2, A_2, E_2) and refer these as TSS_1 and TSS_2. These two beams are TSS as each beam has infinite degrees of freedom. In that case, two ends of the beams must correspond to each other. Hence, corresponding locations along the length of the beams will be obtained by mapping coordinate x_1 of TSS_1 onto coordinate x_2 of TSS_2. This is given by the relation $x_2 = (L_2/L_1)x_1$. The expression for U_{12} at the location $x = x_2$ of TSS_2 is given by

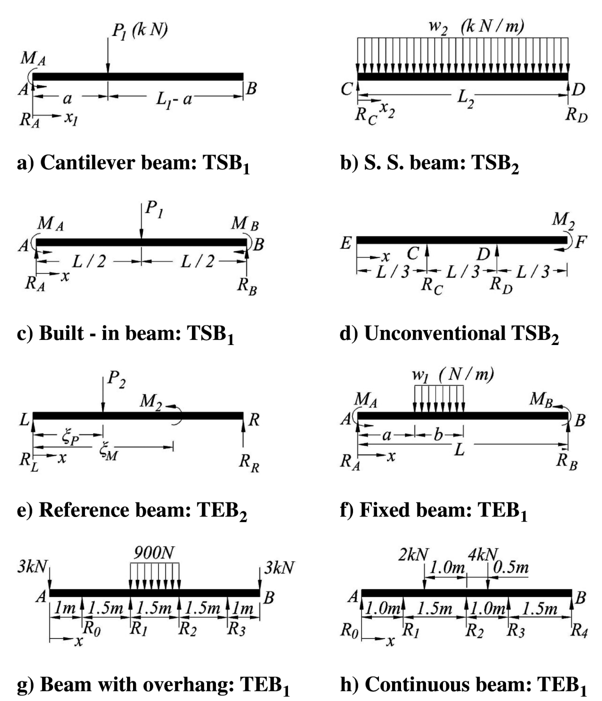

Fig. 3 TSB.

$$U_{12} = \int_{A_1}\int_0^{L_2}\left\{\sigma\left(\frac{L_2}{L_1}x_1\right)\mathrm{d}A_1\right\}_1\{\varepsilon(x_2)\mathrm{d}x_2\}_2$$
$$= \int_{A_1}\int_0^{L_2}\left\{M\left(\frac{L_2}{L_1}x_1\right)\frac{y_1}{I_1}\mathrm{d}A_1\right\}_1\left\{M(x_2)\frac{y_2}{E_2I_2}\mathrm{d}x_2\right\}_2 \tag{2}$$

Where, M is the bending moment expression. Introducing the nondimensional parameter $\zeta = x_1/L_1 = x_2/L_2$, Eq. (2) takes the form

$$U_{12} = \int_0^1\{M(L_2\zeta)\}_1\{M(L_2\zeta)\}_2 L_2\,\mathrm{d}\zeta\int_{A_1}\frac{y_2}{E_2I_2}\frac{y_1}{I_1}\,\mathrm{d}A_1 \tag{3}$$

To have a meaningful evaluation of the integral over A_1, it is prudent to choose beams with equal depth. In such a case corresponding points in the two beams will have same y coordinate (i.e., $y_2 = y_1$) and this integral evaluates to $1/E_2I_2$. The expression for U_{12} in Eq. (3) simplifies to

$$U_{12} = \int_0^1\frac{L_2}{E_2I_2}\{M(L_2\zeta)\}_1\{M(L_2\zeta)\}_2\mathrm{d}\zeta = \int_0^1\frac{L_2}{E_2I_2}\{M^*\}_1\{M_2\}\mathrm{d}\zeta \tag{4}$$

$$U_{21} = \int_0^1\frac{L_1}{E_1I_1}\{M\}_1\{M^*\}_2\mathrm{d}\zeta = \int_0^1\frac{L_2}{E_1I_1}\{M^*\}_1\{M\}_2\mathrm{d}\zeta \tag{5}$$

where $\{M^*\}_1$ corresponds to the location of $\{M\}_2$.

Hence, any two beams are topologically similar as long as their depth is same. For topological similarity loading, boundary conditions and parameters (L, E, and I) of the two beams may be different. Beams that are topologically similar will henceforth be referred to as topologically similar beams (TSBs). If the two beams are of equal length then

$$U_{mn} = \int_0^L\frac{1}{E_nI_n}\{M\}_m\{M\}_n\mathrm{d}x \tag{6}$$

TSBs with the same values for the parameters E, I will be called topologically equivalent beams (TEBs). From Eq. (6) these beams have $U_{12} = U_{21}$ and offer an additional advantage as PQW offers six equations (viz., $W_{12} = U_{12} = U_{21} = W_{21}$) instead of usual two equations ($W_{12} = U_{12}$ and $U_{21} = W_{21}$) for their analysis. Quasi-strain energy for these beams as given by Eqs. (4) and (5) reduces to

$$U_{12} = U_{21} = \int_0^1\frac{L_2}{EI}\{M^*\}_1\{M\}_2\mathrm{d}\zeta \tag{7}$$

and if $L_1 = L_2$ then

$$U_{12} = U_{21} = \int_0^1\frac{1}{EI}\{M\}_1\{M\}_2\mathrm{d}\zeta \tag{8}$$

IV. Validation of Principle of Quasi Work for Beams

PQW is validated by taking the following two beams. TSB_1, shown in Fig. 3a, is a cantilever beam built in at the left end A and free at the right end B, with concentrated load P_1 kN acting at a distance of a meters from end A and having parameters L_1, E_1, I_1. Let the reactions at support A be denoted by R_A and M_A.

TSB_2, shown in Fig. 3b, is a simply supported beam CD with a uniformly distributed load w_2 kN/m, having parameters L_2, E_2, I_2. Let the reactions at supports C and D be denoted by R_C and R_D, respectively.

Quasi energies U_{12} and U_{21} are calculated by substituting $M_1^* = -P_1L_2(\zeta - \frac{a}{L_1})$ and $M_2 = -\frac{w_2L_2^2}{2}(\zeta - \zeta^2)$ in Eqs. (4) and (5), resulting in

$$U_{12} = \frac{w_2P_1}{24E_2I_2}\frac{L_2^4}{L_1^4}a^3(a - 2L_1) \tag{9}$$

and

$$U_{21} = \frac{w_2P_1}{24E_1I_1}\frac{L_2^4}{L_1^4}a^3(a - 2L_1) \tag{10}$$

Quasi work W_{12} is given by

$$W_{12} = \{M_A^*\}_1\{v'(0)\}_2 + \{R_A\}_1\{v(0)\}_2 + \{-P_1\}_1\{v(aL_2/L_1)\}_2$$
$$= \frac{w_2P_1a^3}{24E_2I_2}\frac{L_2^4}{L_1^4}(a - 2L_1) \tag{11}$$

Similarly, W_{21} for the given set of TSB is given by

$$W_{21} = \{R_C\}_2\{v(0)\}_1 + \{R_D\}_2\{v(L_1)\}_1 + \int_0^1\{-w_2\}_2\{v(\zeta)\}_1L_2\mathrm{d}\zeta$$
$$= \frac{w_2P_1a^3}{24E_1I_1}\frac{L_2^4}{L_1^4}(a - 2L_1) \tag{12}$$

Equations (9), (11), (10), and (12) yield $U_{12} = W_{12}$ and $U_{21} = W_{21}$, respectively. This completes the validation of PQW. With this, the link between two beams that are topologically similar is established. This link will prove to be a useful tool for experimental verification of beam designs.

In the examples that follow, length of chosen TSB_2/TEB_2 is taken equal to that of given beam (TSB_1/TEB_1), as it will do away with the mapping of lengths. Upward loads, upward deflections, anticlockwise moments, and anticlockwise rotations are taken as positive.

V. Application of Principle of Quasi Work

In this section, PQW is applied to solve for redundant reactions of a beam AB built in at both the ends, with parameters E_1, I_1, and L and carrying a load P_1 acting at the center as shown in Fig. 3c. The beam has two end moment reactions (M_A and M_B) and two vertical reactions (R_A and R_B) at the supports, making a total of four unknowns. The degree of indeterminacy of the beam is two. Because of symmetry $R_A = R_B = P/2$ and $M_A = M_B$, hence, only one unknown (M_A or M_B) is calculated by using PQW. This given beam is designated as TSB_1.

To illustrate the broader range of applicability of PQW, TSB_2 is chosen in such a way that it should not look like relaxing of constraints (i.e., boundary conditions) as is the usual procedure in conventional methods. Hence, as shown in Fig. 3d, a simply supported beam EF of length L with overhangs of length $L/3$ on both the sides of supports C and D is chosen as TSB_2. Loading on this beam is taken as a concentrated moment M_2 applied at the right-hand free end F. The beam parameters are taken as E_2, I_2, and L. Quasi work W_{12} is calculated as follows:

$$W_{12} = \{R_A\}_1\{v(0)\}_2 + \{M_A\}_1\{v'(0)\}_2$$
$$+ \{-P_1\}_1\{v(L/2)\}_2 + \{R_B\}_1\{v(L)\}_2$$
$$+ \{-M_B\}_1\{v'(L)\}_2 = \frac{M_2L(8M_A - P_1L)}{16E_2I_2} \tag{13}$$

$$W_{21} = \{R_C\}_2\{v(L/3)\}_1 + \{R_D\}_2\{v(2L/3)\}_1$$
$$+ \{-M_2\}_2\{v'(L)\}_1 = 0 \tag{14}$$

As $\{R_C\}_2 = -\{R_D\}_2$, $\{v(L/3)\}_1 = \{v(2L/3)\}_1$ and $\{v'(L)\}_1 = 0$. U_{12} and U_{21} are obtained with the help of Eq. (6) and are equal to

$$U_{12} = \frac{M_2L}{16E_2I_2}(8M_A - P_1L) \tag{15}$$

and

$$U_{21} = \frac{M_2L}{16E_1I_1}(8M_A - P_1L) \tag{16}$$

For TSB, the PQW relation $W_{12} = U_{12}$ [Eqs. (13) and (15)] yields an identity, whereas the relation $W_{21} = U_{21}$ [Eqs. (14) and (16)] yields value for $M_A = P_1L/8$. For TEB, two of the additional four relations [viz., $W_{12} = U_{21}$, Eqs. (13) and (16) and $U_{12} = U_{21}$, Eqs. (15) and (16)] lead to identity, whereas the remaining two relations [viz., $W_{21} = U_{12}$, Eqs. (14) and (15) and $W_{12} = W_{21}$, Eqs. (13) and (14)] yield the solution for M_A.

It follows that unless out of the two quasi energies (U_{12} and U_{21}) and two quasi works (W_{12} and W_{21}) at least one of these is equal to zero, the problem at hand (represented by TSB_1/TEB_1) can not be solved for the unknown(s).

As choosing TSB_2/TEB_2 is in our hands, hence, loading and boundary conditions of the chosen beam should lead to either U_{21} or W_{21} equal to zero. However, it is difficult to visualize such loading and boundary conditions for U_{21}, whereas it is a very simple task to make such a choice for W_{21}. Moreover, calculating quasi work is much easier compared with the evaluation of quasi energy; hence, making use of the equation $W_{12} = W_{21}$ will be a better choice and for that the chosen beam should be a TEB (i.e., having same parameter EI) of the given beam.

It is evident that the solution for the deflection curve of a properly chosen beam along with its loading and boundary conditions is all that is needed to solve the given problem at hand. Hence, we must explore the possibility of developing a beam with properly chosen loading that can be used to solve all types of given beam problems. That such a beam exists will be shown subsequently. With this beam one can keep equations for deflection and slope of its elastic line in the pocket and solve any indeterminate problem for reactions/deflections wherever and whenever one wishes. It is also possible to develop a general purpose interactive computer program that will first generate the requisite number of simultaneous equations (equal to the degree of indeterminacy) and then solve these to obtain values of the chosen redundant reactions. Such a beam (TSB_2) is developed and will be named RB.

VI. Reference Beam

Here, an attempt has been made to define such a beam. RB is defined as a simply supported beam with L and EI as parameters. It carries a point load P_2 at a distance of ξ_p and a concentrated moment M_2 at a distance ξ_m from the left pinned end as shown in Fig. 3e. The equation for deflection of its elastic line is given by

$$6LEIv(x) = M_2x(x^2 + 2L^2 + 3\xi_m^2 - 6L\xi_m) - 3LM_2\langle x - \xi_m\rangle^2 - P_2L\langle x - \xi_p\rangle^3 + P_2x(L - \xi_p)(x^2 + \xi_p^2 - 2\xi_pL) \quad (17)$$

In the next section, use of this reference beam is illustrated by taking beams with different boundary conditions.

VII. Redundant Reactions of Indeterminate Beams

In this section redundant reactions of the indeterminate beams with different support conditions are evaluated by using RB. It has three illustrations, one each for a beam with fixed supports, a beam with overhangs, and a continuous 3 deg indeterminate beam.

A. Illustration 1: Fixed–Fixed Beam

A fixed–fixed beam AB, shown in Fig. 3f, is loaded with a uniform load of intensity w_1 N/m from a meters to $a + b$ meters from the left end A. This beam will be denoted by TEB_1. Let its support reactions be R_A, M_A at the left support A and R_B, M_B at the right support B. As the given beam has two degrees of indeterminacy, two topologically similar beams (TEB_2 and TEB_3) will be chosen to generate two equations for the two chosen unknowns M_A and M_B.

To derive TEB_2 from RB, choose $P_2 = 0$, $\xi_m = 0$, and for TEB_3 choose $P_2 = 0, \xi_m = L$. Using these values in Eq. (17) of RB (TEB_2), the following expressions for W_{12} and W_{13} are obtained:

$$W_{12} = \frac{M_2w}{24LEI}\left[(a+b)^4 - 4L(a+b)^3 + 4L^2(a+b)^2 - a^4 + 4La^3 - 4L^2a^2 - (2M_A - M_B)\frac{4L^2}{w}\right] = W_{21}(=0) \quad (18)$$

$$W_{13} = \frac{M_2w}{24LEI}\left[(a+b)^4 - 2L^2(a+b)^2 - a^4 + 2L^2a^2 + (M_A - 2M_B)\frac{4L^2}{w}\right] = W_{31}(=0) \quad (19)$$

Simultaneous solution of Eqs. (18) and (19) yields:

$$M_A = \frac{w}{12L^2}\{(a+b)^3[3(a+b) - 8L] + a^2(4L - 3a) + 6L^2b(2a+b)\} \quad (20)$$

and

$$M_B = \frac{w}{12L^2}\{(a+b)^3[3(a+b) - 4L] - a^3(4L - 3a)\} \quad (21)$$

This completes the solution process. For end-to-end uniform load (i.e., $a = 0$ and $b = L$) the solution reduces to $M_A = -M_B = wL^2/12$.

B. Illustration 2: Beam with Overhangs

A beam AB with 1 m overhang on each side has four equidistant pinned supports numbered from 0 to 3 starting from the left end, as shown in Fig. 3g. The supports have a separation of 1.5 m. It is loaded by concentrated forces of 3 kN at both ends and by a distributed load of 900 N in its middle span. This is given in Timoshenko and Young [16] as problem 7. The degree of indeterminacy of the beam is two, which reduces to one as the beam is symmetric (i.e., $R_0 = R_3$ and $R_1 = R_2$).

TEB_2 is chosen by taking $\xi_p = 1.5$ m, $L = 4.5$, and $M_2 = 0$ in RB. To calculate W_{21}, end loads of 3000 N are replaced by a statically equivalent force system at the nearest supports 0 and 3. This gives a downward force of 3000 N plus an anticlockwise moment of 3000 Nm acting at support 0, and a downward force of 3000 N plus a clockwise moment of 3000 Nm at support 3. With this

$$W_{12} = (-5357.8125 - 1.5R_1 - 1.3125R_2)P_2/EI = W_{21} = 0 \quad (22)$$

Equation (22) gives $R_1 = R_2 = -1905$ N, which is same as in Timoshenko and Young [16].

C. Illustration 3: Continuous Beam

A four-span continuous beam shown in Fig. 3h is 5 m long and is on five pinned supports numbered from 0 to 4. This beam is chosen for solving the three extra unknown reactions.

Supports are placed at 0, 1, 2.5, 3.5, and 5 m from the left end. It carries a point load of 2 kN at 1.5 m and another point load of 4 kN at 3.0 m from the left end. This is given in Timoshenko and Young [16] as problem 8. The given problem as usual is designated as TEB_1.

In this case PQW is used thrice to generate three simultaneous equations relating reactions R_1, R_2, and R_3 at supports 1, 2, and 3, respectively. Three types of loading on the RB are chosen to yield TEB_2, TEB_3, and TEB_4. TEB_2 is obtained by choosing $L = 5$ m, $M_2 = 0$, and $\xi_p = 1.0$ m, TEB_3 by choosing $L = 5$ m, $M_2 = 0$ and $\xi_p = 2.5$ m, and TEB_4 by choosing $L = 5$ m, $M_2 = 0$ and $\xi_p = 3.5$ m. With these values Eq. (17) yields Eqs. (23–25) for TEB_2, TEB_3 and TEB_4, respectively

$$EI\{v(x)\}_2 = P_2[x(x^2 - 09.0)/7.5 - \langle x - 1.0\rangle^3/6] \quad (23)$$

$$EI\{v(x)\}_3 = P_2[x(x^2 - 18.75)/12. - \langle x - 2.5\rangle^3/6] \quad (24)$$

$$EI\{v(x)\}_4 = P_2[x(x^2 - 22.75)/20 - \langle x - 3.5\rangle^3/6] \quad (25)$$

Expressions for quasi work W_{1n} and W_{n1} for $n = 2$, 3, and 4 are given by

$$W_{1n} = \{R_0\}_1\{v(0)\}_n + \{R_1\}_1\{v(1)\}_n + \{R_2\}_1\{v(2.5)\}_n + \{R_3\}_1\{v(3.5)\}_n + \{R_4\}_1\{v(5)\}_n - \{2000\}_1\{v(1.5)\}_n - \{4000\}_1\{v(3)\}_n \quad (26)$$

$$W_{n1} = \{R_L\}_n\{v(0)\}_1 + \{P_2\}_n\{v(\xi_p)\}_1 + \{R_R\}_n v(5)\}_1 = 0 \quad (27)$$

Equations (26) and (27), in conjunction with Eqs. (23–25), yield the following three equations for $n = 2$, 3, and 4, respectively

$$W_{12} = 1.0666\bar{6}R_1 + 1.4791\bar{6}R_2 + 1.0875R_3 - 08075.0 = W_{21}(=0) \quad (28)$$

$$W_{13} = 1.4791\bar{6}R_1 + 2.6041\bar{6}R_2 + 2.0625R_3 - 13958.\bar{3} = W_{31}(=0) \quad (29)$$

$$W_{14} = 1.80750R_1 + 2.06250R_2 + 1.8375R_3 - 11325.0 = W_{41}(=0) \quad (30)$$

Simultaneous solution of Eqs. (28–30) yields $R_1 = 1343.550$ N, $R_2 = 3110.473$ N, and $R_3 = 1876.756$ N. These values are the same as those given in Timoshenko and Young [16].

VIII. Conclusions

Solving indeterminate beams for redundant reactions by using PQW is easier because of the following:

1. The knowledge of writing bending moment of beams is not required.
2. No integration is required for concentrated loads. Simple multiplication of the given loads of one system with known deflections at the corresponding points of the other system is all that is required.
3. For distributed loads one definite integral is to be evaluated whose integrant is the product of two known expressions (viz., distributed load on the given beam and the equation for the elastic line of RB).
4. No derivatives are required as is the case with conventional energy methods.
5. Words like virtual and complimentary do not exist in PQW as both the systems are real. Hence, no additional effort is needed for learning.
6. The concept of complimentary energy is also absent in PQW.
7. As the equation for deflection curve of RB can be used to solve any kind of beam problem for getting redundant reactions, it is possible to develop an interactive graphic computer package for calculating reactions of beams.

References

[1] Fenves, S. J., and Branin, F. H., "Network Topological Formulations of Structural Analysis," *Journal of Structural Engineering*, Vol. 89, 1963, pp. 483–514.
[2] Sebastian, W. M., "Collapse Considerations and Electrical Analogies for Statically Indeterminate Structures," *Journal of Structural Engineering*, Vol. 130, No. 10, 2004, pp. 1445–1453. doi:10.1061/(ASCE)0733-9445(2004)130:10(1445)
[3] Shai, O., "Deriving Structural Theorems and Methods using Tellegen's Theorem and Combinatorial Representations," *International Journal of Solids and Structures*, Vol. 38, Nos. 44-45, 2001, pp. 8037–8052. doi:10.1016/S0020-7683(01)00153-6
[4] Tellegen, B. D. H., "A General Network Theorem, with Applications," *Philips Research Reports*, Vol. 7, 1952, pp. 259–269.
[5] Tellegen, B. D. H., "A General Network Theorem, with Applications," *Proceedings of the Institute of Radio Engineers*, Vol. 14, 1953, pp. 265–270.
[6] Panditta, I. K., "Some Studies on Computer Aided Model Based Design," Ph.D. Dissertation, Indian Institute of Technology, Bombay, India, 1996.
[7] Penfield, P. Jr., Spencer, R., and Duinker, S., "Tellegen's Theorem and Electrical Networks," Massachusetts Institute of Technology, Research Monograph No. 58, Cambridge, MA, 1970, pp. 1–24.
[8] Argyris, J. H., and Kelsey, S., *Energy Theorems and Structural Analysis*, Butterworths, London, 1960.
[9] Shames, I. A., and Dym, C. L., *Energy and Finite Element Methods in Structural Mechanics*, McGraw–Hill, New York, 1985.
[10] Reissner, E., "Formulations of Variational Theorems in Geometrically Non-linear Elasticity," *Journal of Engineering Mechanics*, Vol. 110, No. 9, 1984, pp. 1377–1390. doi:10.1061/(ASCE)0733-9399(1984)110:9(1377)
[11] Reissner, E., "On Mixed Variational Formulations in Finite Elasticity," *Acta Mechanica*, Vol. 56, Nos. 3–4, 1985, pp. 117–125. doi:10.1007/BF01177113
[12] Akin, J. E., *Finite Elements for Analysis and Design*, Academic Press, London, 1994.
[13] Cook, R. D., Malkus, D. J., and Plesha, M. E., *Concepts and Applications of Finite Element Analysis*, Wiley, New York, 1989.
[14] Panditta, I. K., Shimpi, R. P., and Prasad, K. S. R. K., "On the Theory of Discrete Model Analyses and Design," *International Journal of Solids and Structures*, Vol. 36, No. 16, 1999, pp. 2443–2462. doi:10.1016/S0020-7683(98)00066-3
[15] Munshi, A., and Tikoo, V., "Application of Principle of Quasi Work to the Analysis of Beams," Rept., Jammu, India, 1999.
[16] Timoshenko, S. P., and Young, D. H., "Elements of Strength of Materials," 5th ed., East West Press, New Delhi, India, 1968, pp. 202–246.

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